Math 259A Lecture 19 Notes

Daniel Raban

November 13, 2019

1 Convolvers in $\ell^2(\Gamma)$

1.1 The group von Neumann algebra of $\mathbb{Z}$

Let's give a more concrete description of the elements of $L(\Gamma)$.

Example 1.1. If $\Gamma = \mathbb{Z}$, then $L(\Gamma) \cong L^\infty(\mathbb{T})$ via the Fourier transform. More precisely, $L^\infty(\mathbb{T}) \cong \{\sum c_n z^n \in \ell^2(\mathbb{Z}) : f * g \in \ell^2(\mathbb{Z})\ \forall g \in \ell^2(\mathbb{Z})\}$ via the map $L^\infty(\mathbb{T}) \to L(\mathbb{Z})$ given by $f \mapsto \sum c_n z^n$, where $c_n = \frac{1}{2\pi} \int f e^{-int}\, d\mu$.

It turns out the general picture looks similar to this case.

1.2 Convolver elements in $\ell^2(\Gamma)$

For $\xi \in \ell^2(\Gamma)$, we get $L_\xi : \ell^2 \to \ell^\infty$, where $L_\xi(\eta) = \xi \cdot \eta$. Then $\|L_\xi\|_{\mathcal{B}(\ell^2, \ell^\infty)} \leq \|\xi\|_{\ell^2}$. We also defined $(L_\xi, D(L_\xi))$ as a closed graph operator on ℓ^2, where $D(L_\xi) = L_\xi^{-1}(\ell^2) = \{\eta \in \ell^2 : \xi \cdot \eta \in \ell^2\}$. This domain contains $\mathbb{C}\Gamma$, the finitely supported series, and the operator has closed graph.

Lemma 1.1. *$L_\xi^* = L_{\xi^*}$, where $\xi^*(g) = \overline{\xi(g^{-1})}$.*

Proof. We can show this for monomials, and by linearity, we can show it for all $\eta \in \mathbb{C}\Gamma$. $\square$

Definition 1.1. An element $\xi \in \ell^2(\Gamma)$ is called a **(left) convolver** if $L_\xi(\ell^2) \subseteq \ell^2$ (i.e. $D(L_\xi) = \ell^2(\Gamma)$).

Corollary 1.1. *ξ is a left convolver if and only if ξ^* is a left convolver.*

Proposition 1.1. *If ξ is a convolver, then $L_\xi : \ell^2 \to \ell^2$ is bounded.*

Proof. This follows from the closed graph theorem. $\square$

Lemma 1.2. *If $\xi, \eta, \zeta \in \ell^2(\Gamma)$ and $\xi \cdot \eta, \eta \cdot \eta \in \ell^2$, then $(\xi \cdot \eta) \cdot \zeta = \xi \cdot (\eta \cdot \zeta)$.*

Corollary 1.2. *If ξ, η are convolvers, then $\xi\eta$ is a convolver, and $L_\xi L_\eta = L_{\xi \cdot \eta}$.*

Corollary 1.3. *ξ is a left convolver if and only if ξ^* is a right convolver.*

Proof. $(\xi \cdot \eta)^* = \eta^* \cdot \xi^*$. □

Theorem 1.1. *Let $LC(\Gamma) := \{L_\xi : \xi$ is a convolver$\}$, $RC(\Gamma) := \{R_\xi : \xi$ is a convolver$\}$. Then $LC(\Gamma)$ and $RC(\Gamma)$ are von Neumann algebras. Moreover, $LC(\Gamma) = L(\Gamma) = R(\Gamma)'$, and $RC(\Gamma) = R(\Gamma) = L(\Gamma)'$.*

Remark 1.1. This theorem tends to have limited utility, but it provides great intuition about what $L(\Gamma)$ and $R(\Gamma)$ looks like.

Proof. $LC(\Gamma)$ is SO closed: Let $\{x_i\}$ be left convolvers such that $L_{\xi_i} \xrightarrow{\text{so}} T \in \mathcal{B}(\ell^2)$. Let $|xi = T(\xi_e$. Then $\|\xi_i - \xi\|_{\ell^2} \to 0$ because $\xi_i = L_{\xi_i}(\xi_e) \to T(\xi_e) = \xi$. But also $L_{\xi_i} \to L_\xi$ in $\mathcal{B}(\ell^2, \ell^\infty)$ because $\|L_{\xi_i - \xi}\|_{\mathcal{B}(\ell^2,\ell^\infty)} \leq \|\xi_i - \xi\|_{\ell^2}$. This implies that L_ξ in $\mathcal{B}(\ell^2, \ell^\infty)$. So ξ is a convolver.

We now have that $LC(\Gamma)$ is a SO-closed *-algebra in $\mathcal{B}(\ell^2(\Gamma))$. So it is a von Neumann algebra. We also have $LC(\Gamma) \supseteq \mathbb{C}\Gamma$, the finitely supported convelvers. So $LC(\Gamma) \supseteq L(\Gamma)$; similarly, $RC(\Gamma) \supseteq R(\Gamma)$. Also, we have $LC(\Gamma)$ commutes with $RC(\Gamma)$: $(\xi \cdot \eta) \cdot \zeta = \xi \cdot (\eta \cdot \zeta)$ gives us $R_\eta(L_\xi(\eta)) = L_\xi(R_\xi(\eta))$.

Thus, $L(\Gamma) \subseteq LC(\Gamma) \subseteq RC(\Gamma)'$ and $R(\Gamma) \subseteq RC(\Gamma) \subseteq LC(\Gamma)'$. This implies that $L(\Gamma)' \supseteq LC(\Gamma)' \supseteq RC(\Gamma)$ and $R(\Gamma)' \supseteq RC(\Gamma)' \supseteq LC(\Gamma)$. We claim that $R(\Gamma)' \subseteq LC(\Gamma)$; this will finish the proof.

Let $T \in R(\Gamma)'$ and let $\xi = T(\xi_e)$. Then

$$T(\xi_g) = T(R_{xi_g}(\xi_e) = R_{\xi_g}(T(\xi_e)) = R_{\xi_g}(\xi) = L_\xi(\xi_g).$$

By linearity, $T = L_\xi$ on $\mathbb{C}\Gamma$. These coincide on a dense subset of $\ell^2(\Gamma)$, so $T = L_\xi$. □

Now we will switch our notation. We will denote $L(\Gamma) = \{\sum c_g n_g : \text{square summable}\}$ endowed with the formal product of series. This is to make the connection with Fourier series more apparent. What does the trace state look like with this notation?

$$\tau\left(\sum c_g u_g\right) = c_e.$$

Notice that

$$\langle x, y \rangle = \tau(y^* x) = \langle x, y \rangle_{\ell^2(\Gamma)}.$$

If we let $M = L(\Gamma)$ with this inner product, then $\ell^2(\Gamma) = \overline{M}^{\|\cdot\|_\tau}$ by the GNS construction.

Next time, we will prove the following theorem in two different ways.

Theorem 1.2. *$L(\mathbb{F}_n) \not\cong L(S_\infty)$ for $n \geq 2$.*